

TANGENT BUNDLES OF HYPERBOLIC SPACES AND PROPER AFFINE ACTIONS ON L^p SPACES

INDIRA CHATTERJI, FRANÇOIS DAHMANI, THOMAS HAETTEL AND
JEAN LÉCUREUX

INTRODUCTION

Let G be a locally compact, second countable group, and V be a normed vector space. The group of affine isometries of an affine space of linear part V , is isomorphic to $V \rtimes \overrightarrow{\text{Isom}}V$, where $\overrightarrow{\text{Isom}}V$ is the group of linear isometries of V . Thus, an isometric affine action of G on V gives, by quotient, a representation $\lambda : G \rightarrow \overrightarrow{\text{Isom}}V$, and, by cancellation with a section, a map $c : G \rightarrow V$ satisfying the *cocycle condition* for λ , namely

$$c(gh) = c(g) + \lambda(g)c(h)$$

for all $g, h \in G$. Conversely, a representation $\lambda : G \rightarrow \overrightarrow{\text{Isom}}V$ and a cocycle $c : G \rightarrow V$ produce an affine isometric action of G on V (seen as affine space) by the formula

$$gv = \lambda(g)v + c(g).$$

The cocycle c is a *coboundary*, meaning that it satisfies $c(g) = d - \lambda(g)d$ (for some $d \in V$ and any $g \in G$) if and only if there is a fixed vector in V for G (in which case, it is that vector d). The cocycle is proper (meaning that $\|c(g_n)\| \rightarrow \infty$ if $g_n \rightarrow \infty$ in the sense that it leaves every compact subset of G) if and only if the action is metrically proper (meaning that for all v , $\|g_nv\| \rightarrow \infty$ if $g_n \rightarrow \infty$).

For V a Hilbert space, the existence of a proper isometric action of G on V is the *Haagerup property* for G , and, if G is not compact, is a fierce negation of *property (T)*, since the latter is equivalent to the existence of a fixed point for any action on a Hilbert space.

In this note we are interested in the case where V is an L^p -space for $p > 1$. A proper action of G on an L^p -space is a weakening of a too strong rigidity property of the type of property (T). A contrario, according to [BFGM07], the lattices $SL_n(\mathbb{Z})$, for $n \geq 3$, have a fixed point property for all actions on L^p -spaces (for $1 < p < \infty$).

Discrete countable groups acting properly discontinuously on CAT(0) cubical complexes do act properly on a Hilbert (or L^2) space [NR97,

NR98]. All hyperbolic groups do act properly on some L^p -space, for some $p > 1$ [Yu05, AL17, Nic13], whereas there are hyperbolic groups with property (T) (for instance lattices in $Sp(n, 1)$ according to Kostant [Kos69]). Fisher-Margulis according to [BFGM07] show that if a group has property (T), then there is an $\epsilon > 0$ such that this group has a fixed point for any action on an L^p and for all $p \in [2, 2 + \epsilon)$.

Let us illustrate these results in the case of G a free group. Let \mathcal{E} be the set of oriented edges of a tree on which G acts freely and let $V_1 = \ell^2(\mathcal{E}, \mathbb{R})$. The representation λ is induced by the action of G on the tree. Let d be the vector of $\ell^\infty(\mathcal{E}, \mathbb{R})$ defined by $d(e) = 1$ if e points toward 1 in the tree (meaning that the end of e is closer to 1 than the origin of e), and $d(e) = 0$ otherwise (note that d is not in V_1 as it takes value 1 on an infinite set). Then $c(g) = d - \lambda(g)d$ is a map a priori from G to $\ell^\infty(\mathcal{E}, \mathbb{R})$. It is a coboundary, hence a cocycle. But the support of $c(g)$ is the set of edges in the oriented segments $[1, g]$ and $[g, 1]$. These supports are finite, and therefore c takes its values in V_1 . One checks that $\|c(g)\|$ grows like the square root (because we are in ℓ^2) of the length of the interval $[1, g]$ in the tree, hence the cocycle c is proper.

Now, let $V_2 = \ell^2(G, \ell^1(G, \mathbb{R})) \subseteq \ell^2(G \times G, \mathbb{R})$. If the tree on which G acts is the Cayley tree, the representation λ is the natural one, extending to the diagonal action of G on $G \times G$. We define a vector $d \in \ell^\infty(G, \ell^1(G, \mathbb{R}))$ as the map that, to $h \neq 1$ associates the Dirac mass on the neighbor of h closest to 1, and $d(1)$ is the null function. One notices that under the natural embedding of $\mathcal{E} \subseteq G \times G$, this d extends by 0 the vector constructed above. Again, $d \in \ell^\infty(G, \ell^1(G, \mathbb{R}))$, but d is not in V_2 . Defining $c(g) = d - \lambda(g)d$, one gets a cocycle from G to a priori $\ell^\infty(G, \ell^1(G, \mathbb{R}))$, but that actually takes values in V_2 since $c(g)$ has finite support for all g . Indeed, for a fixed $h \in G$, the map $c(g)(h) : G \rightarrow \mathbb{R}$ is null everywhere except possibly on the two neighbors of h that are respectively closer to 1 and to g , on which it takes values 1 and -1 , or 0 if these points are equal. Hence for g fixed, the map $c(g)(\cdot) : G \rightarrow \mathbb{R}$ is finitely supported on the interval $[1, g]$ in the tree and has ℓ^2 norm proportional to the distance between 1 and g , showing that c is in fact a proper cocycle.

This idea has been generalized using a coarse version of "taking the neighbor closest to 1". The constructions involved in [Yu05] and in [AL17] use a coarse geodesic flow from 1 to h and from g to h and the comparison of the arrivals of these flows. In free groups, they arrive exactly at the same point if h is not on the segment $[1, g]$, but in hyperbolic groups, there could be a difference. Still, this difference is exponentially small in the Gromov product $(1|g)_h$, allowing to adjust the exponent p to beat the growth of the group.

In all these cases, one has to evaluate "how much" an element h should be thought of as being between 1 and g . In both cases, one estimates the difference between how one does see 1 from h and how one does see g from h .

In this note, we cast this point of view in the language of tangent bundles. We define a suitable notion of a *tangent space* for a metric measured space, endowed with a group action, and give a definition of negative curvature for such a tangent space. We then prove that properties of this tangent space give rise to an action of our group on some L^p space, which is proper under some suitable assumptions.

A Radon measure μ on a locally compact geodesic hyperbolic space X is *non-collapsing* if there exists two constants $C \geq 0$, and $v > 0$, such that for all $x \in X$, one has that $\mu(B(x, C)) \geq v$. For instance, the counting measure on a locally finite graph is non-collapsing, and the volume form on a simply connected complete Riemannian manifold with sectional curvature ≤ -1 is non-collapsing, by Gunther's inequality (see [GHL04, Theorem 3.101 (ii)]).

Recall that the volume entropy of such a measure μ is the exponential growth rate of the μ -measure of balls in X (see Definition 2.1).

Theorem 0.1. *Let $\delta \geq 0$ and X be a locally compact geodesic δ -hyperbolic space, and let μ denote a non-collapsing Radon measure on X . Let $h \geq 0$ denote the volume entropy of μ . Assume that a second countable locally compact group G acts properly by isometries on X , preserving μ . Then for any $p > \max(1, h\delta/\log(2))$, G acts properly by affine isometries on a L^p -space.*

One cannot hope to strengthen this result to encompass acylindrical actions: indeed Minasyan and Osin in [MO19] proved that some acylindrically hyperbolic groups have a fixed point property for all L^p -spaces. The following corollary applies for instance to a geometrically finite group action. Observe that these groups are relatively hyperbolic with virtually nilpotent parabolic subgroups.

Recall that a Hadamard manifold is a complete simply connected Riemannian manifold, with non-positive sectional curvature. Its curvature is α -pinched negative, where $0 < \alpha \leq 1$, if it takes value in an interval $[\kappa, \alpha\kappa]$ with $-\infty < \kappa \leq \alpha\kappa < 0$.

Corollary 0.2. *Let M be a α -pinched negative curvature Hadamard manifold of dimension n , and let G be a locally compact second countable group acting properly discontinuously and by isometries on M . Then, for any $p > \frac{n-1}{\sqrt{\alpha}}$, G acts properly on an L^p -space.*

Our notion of tangent bundle also gives an alternate proof of the following result.

Corollary 0.3 (Cornulier-Tessera-Valette [dCTV08]). *Let G be a simple algebraic group of rank 1 over \mathbb{R} or \mathbb{C} , and let δ_H denote the Hausdorff dimension of the visual boundary of the symmetric space X of G . Then for any $p > \max(1, \delta_H)$, G has a proper affine action on some L^p -space.*

Acknowledgments. We are grateful to Mikaël de la Salle for useful discussions, and to the anonymous referee for their comments improving this note.

1. TANGENT BUNDLE ON A METRIC SPACE

In this section we define a tangent bundle in the general setting of a metric measured space.

Definition 1.1. Let (X, d, μ) be a locally compact metric measured space, where μ is a Radon measure. We say that X has a *tangent bundle* TX if:

- (1) TX is a Polish space, with a Borel map $\pi : TX \rightarrow X$
- (2) for every $a \in X$, the fiber $\pi^{-1}(a)$ is a Banach space, denoted T_aX
- (3) there is a measurable map $X \times X \rightarrow TX$, $(a, x) \mapsto \vec{ax}$, so that $\vec{ax} \in T_aX$ for all a and all x , and so that $\vec{aa} = 0$ for all a .

For $\kappa \leq 0$, we say that the tangent bundle has *curvature at most κ* if for every $C \geq 0$, there exists $D_C \geq 0$ such that, for each $a, x, y \in X$ with $d(x, y) \leq C$, we have

$$\|\vec{ax} - \vec{ay}\| \leq D_C e^{\kappa d(a,x)}.$$

We say that the tangent bundle is *proper* if there exists a proper function $f : [0, +\infty) \mapsto [0, +\infty)$ such that, for every $x, y \in X$, for every $p > 1$, we have

$$\int_{a \in X} \|\vec{ax} - \vec{ay}\|^p d\mu(a) \geq f(d(x, y)).$$

If a group G acts by measure preserving isometries on X , we say that TX is *G -equivariant* if, for every a and $g \in G$, there is an isometry $\phi_g : T_aX \rightarrow T_{ga}X$ such that for every $x \in X$, we have

$$\phi_g(\vec{ax}) = \overrightarrow{(ga)(gx)}.$$

Example 1.2. Let M be a uniquely geodesic Riemannian manifold, and TM its usual tangent bundle. For $a \neq x$ in M , define the vector

\vec{ax} in T_aM to be the unit vector tangent to the geodesic from a to x . This data endows M with a tangent bundle in the sense above. If the sectional curvature of M is at most κ , then the tangent bundle has curvature at most κ . Note that contrary to what the notation may suggest, in this example, the vector has norm 1, regardless of the distance between a and x . That tangent bundle is $\text{Isom}(X)$ -equivariant.

Example 1.3. Let Y be a countable simplicial tree, with the graph metric and the counting measure. For every vertex v , we set T_vY to be $\ell^2(\text{Lk}(v), \mathbb{R})$, where $\text{Lk}(v)$ is the set of vertices adjacent to v . Let TY be the disjoint union of the spaces T_aY . For $a \neq x$ in Y , let \vec{ax} be the indicator of the neighbor of a closest to x . This endows Y with a $\text{Isom}(Y)$ -equivariant proper tangent bundle with curvature at most κ , for all κ .

Example 1.4 (Alvarez-Lafforgue [AL17]). Let X be a hyperbolic graph, uniformly locally finite (that is, all balls of fixed radius have a uniform bound on their cardinality), with its graph metric and counting measure. Let δ be a hyperbolicity constant, and for all $a \in X$, let T_aX be the Hilbert (euclidean) space of maps from the ball $B(a, 4\delta)$ of radius 4δ around a , to \mathbb{R} . Let TX be the disjoint union of the T_aX , and for each $x \neq a$, let $\vec{ax} \in T_aX$ be the map $\mu_x(a) : B(a, 4\delta) \rightarrow \mathbb{R}$ constructed by Alvarez and Lafforgue in [AL17, Théorème 4.1]. This endows X with an $\text{Isom}(X)$ -equivariant proper tangent bundle with negative curvature. In this construction, the vector \vec{ax} has unit norm for the ℓ^1 -norm on T_aM .

According to [CD18] this result remains true if one relaxes the assumptions to the case of hyperbolic graphs that are possibly not locally finite, but that are *uniformly fine*, allowing coned-off graphs of relatively hyperbolic groups. We will explain how hyperbolic spaces with controlled growth admit proper tangent bundles of negative curvature in Section 3.

Definition 1.5. Let $1 \leq p \leq \infty$. The tangent bundle TX is said to be *p-uniform* if there exists a measure space Ω and a Borel isomorphism between TX and $X \times L^p(\Omega)$, such that π corresponds by this isomorphism to the first projection.

Definition 1.6. Let (X, d, μ) be a locally compact metric measured space, and assume that X has a tangent bundle TX . For $1 \leq p \leq \infty$, we define $L^p(X, TX)$ as the set of measurable functions $f : X \rightarrow TX$

such that $f(x) \in T_x X$ for every $x \in X$ and

$$\|f\|_p := \left(\int_X \|f(x)\|^p d\mu(x) \right)^{\frac{1}{p}} < \infty.$$

Remark 1.7. If TX is p -uniform, then there exists a measure space Z such that $L^p(X, TX) = L^p(Z)$. Indeed in that case we get, up to isometry, that for all x , $T_x X = L^p(\Omega)$, and therefore, by the previous definition, $L^p(X, TX) = L^p(X, L^p(\Omega)) = L^p(X \times \Omega)$, the last equality being Fubini's theorem.

2. ACTIONS ON L^p -SPACES FROM NEGATIVELY CURVED TANGENT BUNDLES

We now to explain how tangent bundles of negative curvature relate to actions on L^p -spaces.

Definition 2.1. Let (X, d, μ) be a locally compact metric measured space. The *volume entropy* of X is the number h_{vol} defined as the limit

$$h_{vol} = \limsup_{r \rightarrow \infty} \frac{\log \mu(B(x, r))}{r}.$$

The volume entropy is independent of x since eventually, as r grows the ball $B(x, r)$ will contain any other point of X .

Theorem 2.2. *Assume that a second countable locally compact group G acts on a locally compact metric measured space (X, d, μ) of finite volume entropy by measure-preserving isometries. If X has a G -equivariant tangent bundle with curvature at most $\kappa < 0$, then for any $p > \frac{h}{|\kappa|}$, the group G admits an affine action on $L^p(X, TX)$, which is proper when the tangent bundle is proper.*

Proof. For $p > \frac{h}{|\kappa|}$, let $V = L^p(X, TX)$. We define a linear isometric action π of G on V by

$$\pi_g(f)(x) = \phi_g(f(g^{-1}x)).$$

Note that $\pi_g(f)(x) \in \phi_g(T_{g^{-1}x} X) = T_x X$ for any $g \in G$ and $f \in V$. We then fix a basepoint $o \in X$, and set $f_o : X \mapsto TX, x \mapsto \vec{x}o \in T_x X$, so that we can define a cocycle by

$$\begin{aligned} c : G &\rightarrow \ell^\infty(X, TX) \\ g &\mapsto f_o - \pi_g(f_o). \end{aligned}$$

Indeed, since $\pi_g(f_o)(x) \in T_x X$, the map c is well-defined, and it is a cocycle since it is a coboundary. We now prove that c is integrable, i.e.

takes values in V . Fix a $g \in G$ and let $C = d(o, go)$. We have

$$\begin{aligned} \|c(g)\|_V^p &= \int_{x \in X} \|c(g)(x)\|_{T_x X}^p d\mu(x) = \int_{x \in X} \|\vec{x}\delta - \overline{xg}\delta\|_{T_x X}^p d\mu(x) \\ &\leq \int_{x \in X} D_C^p e^{p\kappa d(x,o)} d\mu(x) \leq \int_0^\infty D_C^p e^{p\kappa r} e^{hr} dr < \infty, \end{aligned}$$

since $p\kappa + h < 0$.

If we assume furthermore that TX is a proper tangent bundle, we have $\|c(g)\|^p \geq f(d(o, go))$ for some proper function $f : [0, \infty) \rightarrow [0, \infty)$. Since the action of G on X is proper, we deduce that c is proper as well. \square

3. TANGENT BUNDLES ON HYPERBOLIC SPACES

The main result of this section is the construction of a tangent bundle for a δ -hyperbolic space under some mild assumptions.

Proposition 3.1. *Let (X, d) be a locally compact geodesic δ -hyperbolic space, and some $0 < \varepsilon < \frac{\log(2)}{\delta}$. Consider a non-collapsing Radon measure μ on X , with finite volume entropy h . Let G denote a group acting by isometries on X and preserving μ . Then X has a proper, G -equivariant tangent bundle TX with curvature at most $-\varepsilon$. Furthermore TX can be chosen to be p -uniform (see Definition 1.5) for every $1 \leq p < \infty$.*

In order to construct the tangent bundle we need an adaptation of [GdlH90, Proposition 7.10], suggested in [AL18], that we now describe. Similar constructions, to define visual metrics on the boundary, can be found also in [BH99, Proposition 3.21] and [CDP90, Lemme 1.7], but we want to define a metric in the space instead of the boundary, so we adapted the construction. Recall that in a metric space (X, d) , for a base point $a \in X$ and two points $x, y \in X$, the Gromov product is defined by

$$(x|y)_a = \frac{1}{2}(d(x, a) + d(y, a) - d(x, y))$$

and that X is δ -hyperbolic if and only if, for any points $x, y, z \in X$, one has that

$$(x|y)_a \geq \min\{(x|z)_a, (y|z)_a\} - \delta.$$

Proposition 3.2. *Fix a δ -hyperbolic geodesic space X , and choose $\varepsilon, D > 0$ such that $0 < \varepsilon \leq \frac{\log(2)}{\delta + D}$. Let $\alpha = De^{-2D\varepsilon}$ and $\beta = \frac{8}{\varepsilon}$. For any $a \in X$, there exists a pseudo-distance d_ε^a on X , invariant under the isometry group of X , such that*

$$(1) \quad d_\varepsilon^a(x, y) \leq \beta e^{-\varepsilon(x|y)_a} \text{ for all } x, y \in X,$$

(2) $\alpha e^{-\varepsilon(x|y)_a} \leq d_\varepsilon^a(x, y)$ for every $x, y \in X$ with $d(x, y) \geq 2D$.

Proof. Fix $a \in X$, and define, for $x, y \in X$,

$$d_\varepsilon^a(x, y) = \inf_{c, L} \left\{ \int_0^L e^{-\varepsilon d(a, c(t))} dt \right\},$$

where the infimum is taken over all $L \geq 0$ and 1-Lipschitz maps $c : [0, L] \rightarrow X$ such that $c(0) = x$ and $c(L) = y$. From the definition, d_ε^a is symmetric, nonnegative and satisfies the triangle inequality. It may not be definite though, hence d_ε^a is merely a pseudo-distance on X .

(1) We first look at the upper bound for d_ε^a . Fix $x, y \in X$ and a geodesic $c : [0, L] \rightarrow X$ from x to y . Let $T \in [0, L]$ such that $|d(a, c(T)) - (x|y)_a| \leq \delta$. For every $t \in [0, T]$, we have $|d(c(t), a) - (T - t) - d(a, c(T))| \leq \delta$. Therefore

$$\int_0^T e^{-\varepsilon d(c(t), a)} dt \leq e^{\varepsilon \delta} e^{-\varepsilon d(a, c(T))} \int_0^T e^{-\varepsilon(T-t)} dt \leq \frac{1}{\varepsilon} e^{2\varepsilon \delta} e^{-\varepsilon(x|y)_a} \leq \frac{4}{\varepsilon} e^{-\varepsilon(x|y)_a}$$

since $e^{\varepsilon \delta} \leq 2$. Similarly $\int_T^L e^{-\varepsilon d(c(t), a)} dt \leq \frac{1}{\varepsilon} e^{2\varepsilon \delta} e^{-\varepsilon(x|y)_a}$ and we have proved that $d_\varepsilon^a(x, y) \leq \beta e^{-\varepsilon(x|y)_a}$.

(2) Let us now look at the lower bound, that we obtain by induction on $L \geq D$. Precisely, fix $x, y \in X$, with $d(x, y) \geq 2D$, and fix a 1-Lipschitz path $c : [0, L] \rightarrow X$ from x to y , hence $L \geq D$. Let \mathcal{L} be the set of all $L' \in [D, L]$ such that, for any interval $[u, v] \subseteq [0, L]$ with $D \leq v - u \leq L'$, we have $D e^{-2D\varepsilon} e^{-\varepsilon(c(u)|c(v))_a} < \int_u^v e^{-\varepsilon d(a, c(t))} dt$. We will prove that the interval \mathcal{L} is non-empty and both open and closed in $[D, L]$, so that it has to be the whole $[D, L]$, for any $L \geq D$.

We first show that \mathcal{L} is non-empty, by showing that $[D, 2D] \subseteq \mathcal{L}$. When $D \leq L' < 2D$ and $[u, v]$ has length L' , then for all $t \in [u, v]$ we have by triangular inequality

$$\begin{aligned} 2d(a, c(t)) &\leq d(a, c(u)) + d(a, c(v)) + d(c(u), c(t)) + d(c(v), c(t)) \\ &\leq 2(c(u)|c(v))_a + d(c(u), c(v)) + v - t + t - u \\ &< 2(c(u)|c(v))_a + 4D. \end{aligned}$$

Hence

$$\int_u^v e^{-\varepsilon d(a, c(t))} dt > L' e^{-2D\varepsilon} e^{-\varepsilon(c(u)|c(v))_a} \geq D e^{-2D\varepsilon} e^{-\varepsilon(c(u)|c(v))_a}.$$

So $L' \in \mathcal{L}$, and $[D, 2D] \subseteq \mathcal{L}$.

That \mathcal{L} is open in $[D, L]$ is because the condition defining \mathcal{L} is open and the interval $[D, L]$ is compact.

We will now show that \mathcal{L} is closed in $[D, L]$. Fix $L' \in (D, L]$, and assume that $[D, L'] \subseteq \mathcal{L}$, we will prove that $L' \in \mathcal{L}$. Fix an interval $[u, v] \subseteq [D, L]$ of length L' , and let $R = \int_u^v e^{-\varepsilon d(a, c(t))} dt$. Let $w \in [u, v]$ such that $\int_u^w e^{-\varepsilon d(a, c(t))} dt = \int_w^v e^{-\varepsilon d(a, c(t))} dt = \frac{R}{2}$. We know that the intervals $[u, w]$ and $[w, v]$ both have length shorter than L' . If both intervals $[u, w]$ and $[w, v]$ have length greater than D , the induction hypothesis say that $De^{-2D\varepsilon} e^{-\varepsilon(c(u)|c(w))_a} < \frac{R}{2}$ and $De^{-2D\varepsilon} e^{-\varepsilon(c(w)|c(v))_a} < \frac{R}{2}$. By hyperbolicity, $e^{-\varepsilon(c(u)|c(v))_a} \leq 2 \max\{e^{-\varepsilon(c(u)|c(w))_a}, e^{-\varepsilon(c(w)|c(v))_a}\}$, hence we conclude that

$$De^{-2D\varepsilon} e^{-\varepsilon(c(u)|c(v))_a} < R = \int_u^v e^{-\varepsilon d(a, c(t))} dt.$$

Otherwise, one interval, say $[u, w]$, has length smaller than D , and so $d(c(u), c(w)) < D$ since c is 1-Lipschitz. As $(c(u)|c(v))_a \geq (c(w)|c(v))_a - d(c(u), c(w)) > (c(w)|c(v))_a - D$, we deduce that

$$e^{-\varepsilon(c(u)|c(v))_a} < e^{D\varepsilon} e^{-\varepsilon(c(w)|c(v))_a}.$$

Since the length of $[u, w]$ is smaller than D and $[u, v]$ has length at least $2D$, we deduce that $[w, v]$ has length at least D . By the induction hypothesis we have $De^{-2D\varepsilon} e^{-\varepsilon(c(w)|c(v))_a} < \frac{R}{2}$. Thus

$$De^{-2D\varepsilon} e^{-\varepsilon(c(u)|c(v))_a} < e^{D\varepsilon} \frac{R}{2} \leq R = \int_u^v e^{-\varepsilon d(a, c(t))} dt,$$

since $\frac{e^{D\varepsilon}}{2} \leq 1$, as we have assumed that $(D + \delta)\varepsilon \leq \log(2)$. So we have proved that $L' \in \mathcal{L}$, and hence \mathcal{L} is closed in $[D, L]$.

In conclusion, we have $De^{-2D\varepsilon} e^{-\varepsilon(x|y)_a} \leq d_\varepsilon(x, y)$ for all $x, y \in X$ with $d(x, y) \geq 2D$. \square

We can now prove Proposition 3.1. The construction of the tangent bundle uses the pseudo-distances constructed above, and the desired properties rely on the estimates for those pseudo-distances.

Proof of Proposition 3.1. Let $1 \leq p < \infty$. For any $a \in X$, we set $T_a X = L^p(X, \mathbb{R}, \mu)$ and define

$$TX = \bigcup_{a \in X} T_a X = X \times L^p(X, \mathbb{R}, \mu),$$

which we equip with the product topology. Fix $o \in X$, for each $a, x \in X$, define $\vec{a}\hat{x} \in L^\infty(X, \mathbb{R}, \mu)$ by

$$\vec{a}\hat{x}(\xi) = d_\varepsilon^a(x, \xi) \frac{e^{-d(a, \xi)}}{f(d(a, \xi))^{1/p}}$$

where d_ε^a is the pseudo-distance defined in Proposition 3.2 and

$$f : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad r \mapsto \mu(B(o, r))$$

(which is proper because μ is non-collapsing).

We first notice that $\vec{a}\vec{x} \in T_a X$. Indeed, since μ has finite volume entropy, there exists a constant $h' \geq 1$ such that $f(n+1) \leq h'f(n)$ for all $n \in \mathbb{N}$ and we compute, using that $d_\varepsilon^a \leq \beta$ from Proposition 3.2(1)

$$\begin{aligned} \|\vec{a}\vec{x}\|^p &= \int_{y \in X} \vec{a}\vec{x}(y)^p d\mu(y) \leq \int_{y \in X} \beta^p \frac{e^{-pd(a, \xi)}}{f(d(a, \xi))} d\mu(y) \\ &\leq \beta^p \sum_{n=0}^{\infty} \frac{e^{-pn}}{f(n)} (f(n+1) - f(n)) \leq \beta^p \sum_{n=0}^{\infty} (h' - 1)e^{-pn} = \beta^p E < \infty, \end{aligned}$$

where $E = \sum_{n=0}^{\infty} (h' - 1)e^{-pn}$. This gives TX the structure of a tangent bundle to X .

We now show that the curvature of this bundle is at most $-\varepsilon$. For any $a, x, y, z \in X$, we have

$$\begin{aligned} |\vec{a}\vec{x}(z) - \vec{a}\vec{y}(z)| &= |d_\varepsilon^a(x, z) - d_\varepsilon^a(y, z)| \frac{e^{-d(a, z)}}{f(d(a, z))^{1/p}} \\ &\leq d_\varepsilon^a(x, y) \frac{e^{-d(a, z)}}{f(d(a, z))^{1/p}} \end{aligned}$$

hence $\|\vec{a}\vec{x} - \vec{a}\vec{y}\|^p \leq E d_\varepsilon^a(x, y)^p$. Now, according to Proposition 3.2(1), there exists a constant $\beta > 0$ such that $d_\varepsilon^a(x, y) \leq \beta e^{-\varepsilon(x|y)_a}$. So, for any $C \geq 0$ and any $x, y \in X$ such that $d(x, y) \leq C$, we have

$$\|\vec{a}\vec{x} - \vec{a}\vec{y}\| \leq E^{1/p} \beta e^{-\varepsilon(x|y)_a} \leq D_C e^{-\varepsilon d(a, x)}$$

because $(x|y)_a \geq d(x, a) - C$, where D_C is any constant larger than $E^{1/p} \beta e^{\varepsilon C}$. That is, the curvature of TX is at most $-\varepsilon$.

It remains to show that the tangent bundle is proper. We choose C is sufficiently large so that $\alpha - \beta e^{-\varepsilon C} > 0$, where α and β are the constants of Proposition 3.2 and let $C' = 2D + \frac{5C}{2} > 0$, where D is again as in Proposition 3.2. For $x, y \in X$, we define

$$A(x, y) = \{a \in X \mid d(x, a) + d(a, y) \leq d(x, y) + C, d(a, x), d(a, y) \geq C'\},$$

this is the set of points that are almost on a geodesic between x and y , and far enough from x and y (it could be empty if x and y are too close, but we will only need the case when x and y are far apart). For

any $x, y \in X$, assuming that $z \in A(x, a)$, then

$$\begin{aligned} (x|z)_a &= \frac{d(a, x) + C - C + d(a, z) - d(x, z)}{2} \\ &\geq \frac{d(a, z) + d(z, x) - C + d(a, z) - d(x, z)}{2} \\ &= d(a, z) - \frac{C}{2} \geq C' - \frac{C}{2} = 2D + 2C, \end{aligned}$$

so that, with Proposition 3.2(1), we obtain $d_\varepsilon^a(x, z) \leq \beta e^{-\varepsilon(2D+2C)}$. Similarly, again assuming that $z \in A(x, a)$, we have that

$$\begin{aligned} (y|z)_a &= \frac{d(a, y) + d(a, z) - d(y, z)}{2} \\ &\leq \frac{d(a, y) + d(a, x) - d(x, z) + C - d(y, z)}{2} \\ &\leq \frac{d(a, y) + d(a, x) - d(x, y) + C}{2} \leq C. \end{aligned}$$

If we furthermore assume that $a \in A(x, y)$, we then have

$$\begin{aligned} d(y, z) &\geq d(x, y) - d(x, z) \geq d(x, y) - d(x, a) + d(a, z) - C \\ &\geq d(x, y) - d(x, a) - C \geq d(a, y) - 2C \geq C' - 2C \geq 2D. \end{aligned}$$

So with Proposition 3.2(2), we deduce that $d_\varepsilon^a(y, z) \geq \alpha e^{-\varepsilon C}$. Since we get the same estimate when $z \in A(a, y)$, for any $x, y \in X$ and $a \in A(x, y)$, we have

$$\begin{aligned} \|\vec{ax} - \vec{ay}\|^p &= \int_{z \in X} |d_\varepsilon^a(x, z) - d_\varepsilon^a(z, y)|^p \frac{e^{-pd(a, z)}}{f(d(a, z))} d\mu(z) \\ &\geq \int_{z \in A(x, a) \cup A(a, y)} |d_\varepsilon^a(x, z) - d_\varepsilon^a(z, y)|^p \frac{e^{-pd(a, z)}}{f(d(a, z))} d\mu(z) \\ &\geq \int_{z \in A(x, a) \cup A(a, y)} K^p \frac{e^{-pd(a, z)}}{f(d(a, z))} d\mu(z), \end{aligned}$$

where $K = \alpha e^{-\varepsilon C} - \beta e^{-\varepsilon(2D+2C)} \geq e^{-\varepsilon C}(\alpha - \beta e^{-C\varepsilon}) > 0$.

Assume now that $d(x, y) \geq 4C' + 2C$ and take $z \in [a, x]$ such that $d(a, z) = C' + \frac{C}{2}$ and $d(z, x) \geq C' + \frac{C}{2}$ (such an element exists because X is geodesic), one checks that $B(z, \frac{C}{2}) \subseteq A(x, a)$. Moreover, for any $z' \in B(z, \frac{C}{2})$, we have $d(a, z') \leq d(a, z) + \frac{C}{2} = C' + C$. Hence, for any

$a \in A(x, y)$ and z as above, we have

$$\begin{aligned} \|\vec{ax} - \vec{ay}\|^p &\geq \int_{z' \in B(z, \frac{C}{2})} K^p \frac{e^{-pd(a, z')}}{f(d(a, z'))} \geq \int_{z' \in B(z, \frac{C}{2})} K^p \frac{e^{-p(C'+C)}}{f(C'+C)} \\ &\geq \mu(B(z, \frac{C}{2})) K^p \frac{e^{-p(C'+C)}}{f(C'+C)} \geq v K^p \frac{e^{-p(C'+C)}}{f(C'+C)} = K' > 0. \end{aligned}$$

where $v > 0$ is a constant given by the non-collapsing assumption on the measure μ . Hence, for any $x, y \in X$ such that $d(x, y) \geq 4C' + 2C$ and for any $p \geq 1$, we have

$$\int_{a \in X} \|\vec{ax} - \vec{ay}\|^p d\mu(a) \geq \int_{a \in A(x, y)} K' d\mu(a) = \mu(A(x, y)) K'.$$

To conclude that the tangent bundle is proper we only need to see that $\mu(A(x, y))$ grows at least like the distance between x and y . To do that, fix $x', y' \in X$ on a geodesic from x to y such that $d(x, x') = C' + \frac{C}{2}$ and $d(y, y') = C' + \frac{C}{2}$. Since the $\frac{C}{2}$ -neighbourhood of a geodesic $[x', y']$ is contained in $A(x, y)$, we have

$$\mu(A(x, y)) \geq v \left(\frac{d(x', y')}{C} - 1 \right) \geq \frac{v}{C} (d(x, y) - 2C' - 2C).$$

As a consequence, we have for any $x, y \in X$ with $d(x, y) \geq 4C' + 2C$

$$\int_{a \in X} \|\vec{ax} - \vec{ay}\|^p \geq \frac{K'v}{C} (d(x, y) - 2C' - 2C),$$

so the tangent bundle is proper. \square

4. APPLICATIONS

We can now finish the proof of our main result.

Proof of Theorem 0.1. Consider X, G, μ and p as in the statement. By Proposition 3.1, X admits a G -equivariant p -uniform tangent bundle of curvature at most $\log(2)/\delta$. By Theorem 2.2, G admits a proper action on $L^p(X, TX)$, and with Remark 1.7, we get the desired action. \square

Proof of Corollary 0.2. Let n be the dimension of the manifold M and we can assume, by rescaling, that the sectional curvature of M lies in the interval $-1 \leq -\alpha$. We will use comparisons with the hyperbolic space, with its Riemannian structure $(\mathbb{H}^n, g_{\mathbb{H}^n})$. Let μ be the volume form on M . It is non collapsing because the curvature is less than $-\alpha$ (by Gunther's inequality [GHL04, Theorem 3.101 (ii)], as already mentioned). Since the curvature is bounded from below, we can apply Bishop's inequality [GHL04, Theorem 3.101 (i)] to obtain that the volume entropy h of the measure μ is at most that of the n -dimensional

real hyperbolic space, i.e. $h \leq n - 1$. Furthermore, the manifold M has hyperbolicity constant that that of the n -dimensional real hyperbolic space rescaled by $\frac{1}{\sqrt{\alpha}}$, which is $\frac{\log(2)}{\sqrt{\alpha}}$.

Therefore we can apply Theorem 0.1, to conclude that for any $p > \frac{n-1}{\sqrt{\alpha}}$, G acts properly by affine isometries on a L^p space. \square

To finish our proof of Cornulier-Tessera-Valette's result in [dCTV08], recall that the *critical exponent* of an action of a group G on a metric space (X, d) is defined as

$$\delta_c := \limsup_n \frac{\log N(x, r)}{r}$$

where $N(x, r)$ is the cardinal of $\{\gamma \in G \mid d(x, \gamma x) \leq r\}$.

Proof of Corollary 0.3. The Riemannian metric on a rank one symmetric space X for G , is CAT(-1) and defines the Hausdorff dimension δ_H of the visual boundary. We consider μ the volume form of this Riemannian manifold. Since G acts transitively and measure preserving, μ is non-collapsing. According to [Pau97, Theorem 0.2], the critical exponent of any cocompact lattice in X is equal to δ_H , hence the volume entropy of X is equal to δ_H . Let TX be the Riemannian tangent bundle, which has curvature ≤ -1 (see Example 1.2). Note that furthermore TX is Borel-isomorphic to $X \times \mathbb{R}^n$ where $n = \dim(X)$ and hence it is p -uniform for all p . Recall that X is Gromov $\log(2)$ -hyperbolic. Hence according to Theorem 2.2, we deduce that for any $p > \max(1, \delta_H)$, G has a proper affine action on $L^p(X, TX)$. Using Remark 1.7, we get the desired action. \square

REFERENCES

- [AL17] Aurélien Alvarez and Vincent Lafforgue. Actions affines isométriques propres des groupes hyperboliques sur des espaces ℓ^p . *Expo. Math.*, 35(1):103–118, 2017.
- [AL18] Aurélien Alvarez and Vincent Lafforgue. Actions affines isométriques propres des groupes hyperboliques sur des quotients d'espaces ℓ^p . *Ann. Sci. Éc. Norm. Supér. (4)*, 51(6):1389–1398, 2018.
- [BFGM07] Uri Bader, Alex Furman, Tsachik Gelander, and Nicolas Monod. Property (T) and rigidity for actions on Banach spaces. *Acta Math.*, 198(1):57–105, 2007.
- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

- [CD18] Indira Chatterji and François Dahmani. Proper actions on ℓ^p spaces for relatively hyperbolic groups. *arXiv e-prints*, page arXiv:1801.08047, Jan 2018.
- [CDP90] M. Coornaert, T. Delzant, and A. Papadopoulos. *Géométrie et théorie des groupes*, volume 1441 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], With an English summary.
- [dCTV08] Yves de Cornulier, Romain Tessera, and Alain Valette. Isometric group actions on Banach spaces and representations vanishing at infinity. *Transform. Groups*, 13(1):125–147, 2008.
- [GdlH90] Étienne Ghys and Pierre de la Harpe. Panorama. In *Sur les groupes hyperboliques d’après Mikhael Gromov (Bern, 1988)*, volume 83 of *Progr. Math.*, pages 1–25. Birkhäuser Boston, Boston, MA, 1990.
- [GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, third edition, 2004.
- [Kos69] Bertram Kostant. On the existence and irreducibility of certain series of representations. *Bull. Amer. Math. Soc.*, 75:627–642, 1969.
- [MO19] Ashot Minasyan and Denis Osin. Acylindrically hyperbolic groups with exotic properties. *J. Algebra*, 522:218–235, 2019.
- [Nic13] Bogdan Nica. Proper isometric actions of hyperbolic groups on L^p -spaces. *Compos. Math.*, 149(5):773–792, 2013.
- [NR97] Graham Niblo and Lawrence Reeves. Groups acting on CAT(0) cube complexes. *Geom. Topol.*, 1:approx. 7 pp. 1997.
- [NR98] Graham A. Niblo and Martin A. Roller. Groups acting on cubes and Kazhdan’s property (T). *Proc. Amer. Math. Soc.*, 126(3):693–699, 1998.
- [Pau97] Frédéric Paulin. On the critical exponent of a discrete group of hyperbolic isometries. *Differential Geom. Appl.*, 7(3):231–236, 1997.
- [Yu05] Guoliang Yu. Hyperbolic groups admit proper affine isometric actions on l^p -spaces. *Geom. Funct. Anal.*, 15(5):1144–1151, 2005.